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ON A CLASS OF POLYNOMIALS CONNECTED WITH THE KORTEWEG-DE VRIES EQUATION

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ABSTRACT

In [1] special classes of solutions of the Korteweg-de Vries equation

$$u_t = 3uu_x - \frac{1}{2} u_{xxx} = X_2 u$$

were studied, in particular, all those $u = u(x,t)$ which are rational functions of x for each value of t. It turns out that these solutions are rational functions of t as well and of very special structure. In this paper we give a new construction of these solutions with emphasis on their algebraic properties.

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ON A CLASS OF POLYNOMIALS CONNECTED WITH THE KORTEWEG-DE VRIES EQUATION

M. Adler and J. Moser

In [1] special classes of solutions of the Korteweg-de Vries equation

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were studied, in particular, all those $u = u(x,t)$ which are rational functions of x for each value of t. It turns out that these solutions are rational functions of t as well and of very special structure. In this paper we give a new construction of these solutions with emphasis on their algebraic properties.

To describe the family of rational solutions of (1.1), one does well to introduce the sequence of associated Korteweg-de Vries equation

(1.2) $$u_t = X_k u \qquad k = 1,2,\ldots$$

which are related by

$$X_k = \frac{\partial}{\partial x} \frac{\delta H_k}{\partial u}$$

to the sequence of conservation laws

$$H_k = \int P_k(u,u',\ldots)dx$$

associated with (1.1). These X_k can be recursively defined by

$$X_{k+1}(u) = \left(u \frac{\partial}{\partial x} + \frac{\partial}{\partial x} u - \frac{1}{2}\left(\frac{\partial}{\partial x}\right)^3\right) \frac{\delta H_k}{\partial u}$$

as was shown originally by Lenard, see [3]. For the notation used here we refer to [6,1].

The above nonlinear differential operators commute, and so do the flows $e(t_k X_k)$ generated by them. We ask for the manifold M of rational functions invariant under all the flows $e(t_k X_k)$. It is one of the results of [1] that M decomposes into denumerably many manifolds M_d of dimensions d for $d = 1,2,\ldots$ and, moreover, each M_d is generated from the single

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function

$$u = \frac{d(d+1)}{x^2}$$

by the flows $e(\Sigma\, t_k X_k)$ (see [1], Section 3, Theorem 2).

In this paper we want to give a representation of M_d in terms of a class of polynomials $\theta_d = \theta_d(\tau_1,\tau_2,\ldots,\tau_d)$ depending on d variables. These polynomials will be defined recursively in Section 2 and they allow the representation of all $u \in M_d$ in the form

$$(1.3) \qquad u_d(x) = -2\left(\frac{\partial}{\partial x}\right)^2 \log \theta_d(\tau_1 + x, \tau_2, \ldots, \tau_d)$$

and this representation is one to one, so that $\tau_1,\tau_2,\ldots,\tau_d$ can be viewed as global coordinates on M_d. The X_k give rise to vector fields Γ_k on M_d which are expressible in terms of the τ_j, and in Section 5 we will determine these Γ_k. It turns out that the $\tau_2,\ldots,\tau_d$ can be subjected to a group of birational transformations

$$\tau_j^* = a_j\tau_j + g_j(\tau_1,\ldots,\tau_{j-1}); \qquad a_j \neq 0 ,$$

g_j being polynomials, without affecting the above representation. Moreover, the parameters τ_j^* can be introduced so that

$$\Gamma_k = \frac{\partial}{\partial \tau_k^*} ,$$

i.e. that the solutions of $u_t = X_k u$ are given by

$$u_d(x, \tau_2^*, \ldots, \tau_k^* + t, \tau_{k+1}^*, \ldots, \tau_d^*) .$$

In other words the τ_k^* can be identified with t-variable t_k of X_k. This picture was developed already in [1] but here this representation is made more explicit through (1.3) and the construction of the polynomials θ_k. The representation (1.3) is analogue to that of Its and Matveev [4] for the case of solutions of (1.1) having a fixed period in x and for which the corresponding Hill's equation has only finitely many simple eigenvalues.

It is conceivable that (1.3) could be obtained by a limit process from the formula [4], but we did not succeed in this way.* Similarly one may expect that (1.3) could be obtained as the limit of the N-soliton potentials [8], but neither did we succeed in this way.

We mention that the solutions of the type (1.3) for the case $d = 2$ were considered by H. Moses [9].

The construction of the θ_k as well as the proof of the above statements are based on a transformation of differential operators $L = -D^2 + u$ into each other which is certainly not new (see [2,11,12]) and the so-called Miura transformation [7] for which we give a natural derivation. This derivation is based on the factorization

$$L = A^*A; \qquad A = D - v$$

where $u = v' + v^2$. Similarly as in Lax' work [5], where one considers deformations of operators L in the equivalence class of operators of the form $U^{-1}LU$ for unitary U, here we consider deformations of operators A in the equivalence class of operators U_1AU_2, where U_1, U_2 are two unitary operators. If we apply these ideas to formal differential operators we are lead to the modified Korteweg-de Vries equation which by $u = v' + v^2$ is transformed into (1.1). This follows from the fact that any deformation of A of this kind gives rise to an iso-spectral deformation of L. The ideas are explained in Section 3.

On the other hand if $L = A^*A$ one gets a second differential operator $\tilde{L} = AA^* = -D^2 + \tilde{u}$ by exchanging the role of A and A^*. Moreover, $\tilde{L}$ is also iso-spectrally deformed under the above deformation. This gives rise to a Bäcklund transformation of $u = v' + v^2$ into $\tilde{u} = -v' + v^2$ leaving the X_j invariant. Applying this transformation repeatedly we construct the sequence u_d of (1.3). This transformation has also been employed in the construction of N-soliton solutions [11].

In [1] the functions $u_d \in M_d$ were described in terms of their poles $x_1, x_2, \ldots, x_n$, $n = \frac{1}{2}d(d+1)$, which have to satisfy the conditions

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* H. McKean, via personal communication, informed us of his success in carrying out this approach

$$(1.4) \qquad \sum_{\substack{j=1 \\ j \neq k}}^{n} (x_k - x_j)^{-3} = 0 \quad \text{for} \quad k = 1,2,\ldots,n \ .$$

In contrast, here we give preference to the functions

$$\theta_d = \prod_{j=1}^{n} (x - x_j)$$

which are constructed in Section 2. As a consequence of their properties their roots satisfy (1.4) and all solutions of (1.4) can be so obtained. Thus the $\tau_1,\ldots,\tau_d$ can be viewed as uniformizing variables for the algebraic variety (1.4).

The second author wishes to express his thanks for the hospitality of the University of Wisconsin. We are grateful to C. Conley at whose seminar these results were presented and to H. McKean for discussions on this subject. We also are indebted to P. Deift for pointing out the relevance of the factorization of second order operators.

§2. Construction of the polynomials θ_k.

In this section we construct a sequence of polynomials $\theta_k(\tau_1,\ldots,\tau_k)$ for $k = 0,1,\ldots$ of k variables, which will be considered as polynomials of one variable $x = \tau_1$, the others figuring as parameters. As such they have the degree $n_k = \frac{1}{2}k(k+1)$. They are defined recursively by

(2.1) $$\theta_0 = 1,\ \theta_1 = x = \tau_1$$

and the differential equation

(2.2) $$\theta'_{k+1}\theta_{k-1} - \theta_{k+1}\theta'_{k-1} = (2k+1)\theta_k^2 \quad {}^{*})$$

which leaves an integration constant available. We fix this constant by the normalization that the coefficient $x^{n_{k-1}}$ in θ_{k+1} is equal to τ_{k+1}. This defines the polynomials uniquely and at each recursive step one picks up a new integration constant τ.

For the first few polynomials one finds

$$\theta_0 = 1$$
$$\theta_1 = x$$
$$\theta_2 = x^3 + \tau_2$$
$$\theta_3 = x^6 + 5\tau_2 x^3 + \tau_3 x - 5\tau_2^2$$
$$\theta_4 = x^{10} + 15\tau_2 x^7 + 7\tau_3 x^5 - 35\tau_2\tau_3 x^2 + 175\tau_2^3 x - \frac{7}{3}\tau_3^2 + \tau_4 x^3 + \tau_4\tau_2 .$$

However, it is by no means obvious that the above differential equations can be solved within the class of polynomials. That this is the case is the contents of the following proposition which is purely algebraic. Its proof is based on division properties of polynomials and properties of the Wronskian

$$[A,B] = A'B - AB' .$$

Proposition: There exists a unique sequence of polynomials $\theta_k(x,\tau_2,\ldots,\tau_k)$ in k variables with rational coefficients satisfying (2.1), the recursion

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*) prime or D stands for $\frac{\partial}{\partial x}$.

equation (2.2) and the normalization condition mentioned above.

Moreover, the symmetric expression

(2.3) $$\{\theta_k, \theta_{k-1}\} = [\theta_k', \theta_{k-1}] + [\theta_{k-1}', \theta_k]$$

vanishes identically*); { } is defined by this relation.

The θ_k, θ_k' considered as polynomials in x over the ring $Q[\tau_2, \ldots, \tau_k]$ of polynomials in $\tau_2, \ldots, \tau_k$ with rational coefficients have no common factors. Finally, $\theta_k = x^{n_k} + \cdots$ with $n_k = \frac{1}{2} k(k+1)$.

Proof: The above statements will be proved together by induction on k. Assuming they have been proven for $k \leq d$ we set

$$X = \theta_{d-1},\ Y = \theta_d .$$

We aim to solve the recursion equation

(2.4) $$[Z, X] = (2d+1)Y^2$$

for Z. Instead we solve the equation

(2.5) $$[Z, X] = (2d+1)Y^2 + PX ,$$

where P is a polynomial in x of degree $\deg P < \deg X$ and where the coefficient of $x^{n_{d-1}}$ in Z vanishes. Equation (2.5) represents a system of $2n_k + 1$ linear equations for the $n_{k+1} + n_{k-1}$ coefficients of Z and P. From $n_k = \frac{1}{2} k(k+1)$ we derive

$$n_{k+1} + n_{k-1} = 2n_k + 1$$

and the number of equations and unknowns match. Thus the solvability of (2.5) is assured if we show that the homogeneous system has only the trivial solution. Therefore we consider the equation

$$Z'X - ZX' = PX .$$

Setting $Z = cx^{\alpha} + \cdots,\ c \neq 0,$ the highest term on the left hand side is

$$c(\alpha - n_{d-1})x^{\alpha + n_{d-1} - 1}$$

*A similar identity also holds for N-soliton potentials [8].

while the right hand side has terms of order $\leq 2n_{d-1} - 1$ only. Hence

$$\deg Z = \alpha \leq 2n_{d-1} - n_{d-1} = n_{d-1} .$$

By our normalization the coefficient of $x^{n_{d-1}}$ in Z vanishes, hence $\deg Z < \deg X = n_{d-1}$. Moreover,

$$ZX' = (Z' - P)X$$

and since X, X' have no common factor X must divide Z, which is impossible unless $Z = 0$, since $\deg Z < \deg X$. But $Z = 0$ implies $P = 0$ and the solvability of (2.5) for polynomials Z,P with coefficients rational in $\tau_2,\ldots,\tau_d$ is established.

Next we show that $P = 0$ in (2.5) so that Z in fact is a solution of (2.4). Observe that

$$[[Z,X],X'] = [Z',X']X$$

is divisible by X. Thus if we take the Wronskian of both sides of (2.5) with X' we find

$$[Z',X']X = (2d + 1)[Y^2,X'] + [PX,X'] = \tag{2.6}$$
$$= (2d + 1)[Y^2,X'] + [P,X']X + PX'^2 .$$

The crucial observation is that

$$[Y^2,X'] = 2YY'X' - Y^2X'' = -\{Y,X\}Y + YY''X$$

with the notation of (2.3). Thus, since by induction hypothesis, $\{Y,X\} = 0$, we conclude that $[Y^2,X']$ is divisible by X, hence by (2.6) also PX'^2 is divisible by X. This implies $P = 0$ as $\deg P < \deg X$, and since X, X' have no common factor.

Thus we have shown that the polynomial Z is a solution of (2.4), and thus its coefficient of $x^{n_{d-1}}$ is zero. Z is uniquely determined by this requirement. So far the coefficients are known to be rational functions of $\tau_2,\ldots,\tau_d$ and we verify now that actually they are polynomials in $\tau_2,\ldots,\tau_d$. For this purpose we compare coefficients in (2.4) for

$$Z = \sum_\alpha z_\alpha x^\alpha .$$

We find that the terms of degree $\alpha + n_{d-1} - 1$ in (2.4) are

$$(\alpha - n_{d-1}) z_\alpha +$$

plus terms which are polynomials in $z_{\alpha+1}, \ldots, z_{n_{d+1}}$ and the known coefficients of Y and X. Thus we can determine z_α recursively as polynomials in the coefficients of X, Y, with rational coefficients. By induction on d we conclude that the z_α are polynomials with rational coefficients in $\tau_2, \tau_3, \ldots, \tau_d$. The highest coefficient is found for $\alpha = n_{d+1}$ from

$$(n_{d+1} - n_{d-1}) z_\alpha = (2d + 1) .$$

Since $n_{d+1} - n_{d-1} = 2d + 1$ we have $z_\alpha = 1$ for $\alpha = n_{d+1}$. Thus we have

$$Z = x^{n_{d+1}} + \cdots$$

is a polynomial with coefficients z_α in the ring $Q[\tau_2, \ldots, \tau_d]$ of polynomials in $\tau_2, \ldots, \tau_d$.

If we set

$$\theta_{d+1} = Z + \tau_{d+1}\theta_{d-1}$$

then θ_{d+1} is clearly a solution of (2.2) and satisfies the desired normalization condition.

The proof will be complete if we verify (2.3) for $k = d + 1$ and show that θ_{d+1}, θ'_{d+1} have no common factors.

To do this we use the identity

$$[[\theta_{d+1}, X], Y^2] = XY\{\theta_{d+1}, Y\} - \theta_{d+1}Y\{Y, X\} \tag{2.7}$$

where again $Y = \theta_d$, $X = \theta_{d-1}$. By construction

$$[\theta_{d+1}, X] = (2d + 1)Y^2$$

and the left side of (2.7) vanishes. By induction hypothesis $\{Y, X\} = 0$, hence we conclude from (2.7) that

$$\{\theta_{d+1}, Y\} = 0 .$$

Finally we observe that $\theta_1 = x$, $\theta_2 = x^3 + \tau_2$ satisfy the property that θ_k, θ_k' have no common factor. Our assertion is proved inductively for

$$\theta_{d+1} = Z + \tau_{d+1}\theta_{d-1} = Z + \tau_{d+1}X$$

from the following remark: If X, X' have no common factor over $Q[\tau_2,\ldots,\tau_d]$ and Z is any other polynomial then

$$Z + \tau X \quad \text{and} \quad Z' + \tau X'$$

have no common factor over $Q[\tau_2,\tau_3,\ldots,\tau_d,\tau]$. Such a common factor would have to be linear in τ, and one readily shows that its degree in x must be zero.

This completes the proof of the proposition.

Remark 1. The above polynomials have the homogeneity property

$$\theta_k(\lambda\tau_1,\lambda^3\tau_2,\ldots,\lambda^{2k-1}\tau_k) = \lambda^{n_k}\theta_k(\tau_1,\ldots,\tau_k) \ .$$

We say the θ_k are "isobaric" of degree n_k if we assign τ_j the weight $2j - 1$.

Indeed, if we replace x, τ_j, θ_k by

$$x^* = \lambda x, \ \tau_j^* = \lambda^{2j-1}\tau_j, \ \theta_k^* = \lambda^{n_k}\theta_k$$

then one verifies that (2.1), (2.2) and the normalization condition are preserved. Hence by uniqueness

$$\theta_k^* = \theta_k(\tau^*) \ ,$$

proving the remark.

Remark 2: The parameters $\tau_2,\tau_3,\ldots$ were introduced by a rather arbitrary normalization condition. One can free oneself from this arbitrariness by replacing the τ_j by

$$\tau_j^* = a_j\tau_j + g_j(\tau_2,\tau_3,\ldots,\tau_{j-1}) \ ,$$

where g_j are polynomials with rational coefficients and $a_j \neq 0$ is a rational number. Moreover, we require that g_j be isobaric of degree j which amounts to requiring that the transformation $\tau_j \to \tau_j^*$ commutes with $\tau_j \to \lambda^{2j-1}\tau_j$. In fact, the above birational transformations form a group and we will reserve the freedom to pick an appropriate such transformation (see Section 5).

In the following we need another property of these polynomials.

Lemma 1: For fixed $d \geq 1$ let

$$\theta_d(x + \tau_1, \tau_2, \ldots, \tau_d) = x^n + \sigma_1 x^{n-1} + \cdots + \sigma_n, \qquad n = n_d .$$

Then the σ_j are isobaric polynomials in $\tau_1, \ldots, \tau_d$ of degree j. Moreover

$$\sigma_{2j-1} = \alpha_j \tau_j + q(\tau_2, \ldots, \tau_{j-1}) \quad \text{for} \quad j = 1, 2, \ldots, d$$

where $\alpha_j \neq 0$ and q is a polynomial, isobaric of degree $2j - 1$.

Corollary: This lemma implies that the above relation can be solved for $\tau_1, \tau_2, \ldots, \tau_d$ and $\tau_1, \ldots, \tau_d$ expressed as isobaric polynomials, with a non-vanishing linear term, of $\sigma_1, \sigma_3, \sigma_5, \ldots, \sigma_{2d-1}$. Hence $\tau_1, \tau_2, \ldots, \tau_d$, are in birational, isobaric equivalence with $\sigma_1, \sigma_3, \ldots, \sigma_{2d-1}$.

Proof: It is obvious from the above proposition that the σ_j are isobaric polynomials in the $\tau_2, \ldots, \tau_d$ and we just have to verify that $\alpha_j \neq 0$. Of course, $\alpha_j = \alpha_j(d)$ depends on d, and we simply compute it.

Clearly α_j is the coefficient of $\tau_j x^{n-2j+1}$ in $\theta = \theta_d$ and therefore we have

$$\frac{\partial}{\partial \tau_j} \theta = \alpha_j x^{n-2j+1} \quad \text{for} \quad \tau_1 = 0, \ldots, \tau_d = 0$$

while

$$\theta = x^n \quad \text{for} \quad \tau_1 = 0, \ldots, \tau_d = 0 .$$

Hence if we differentiate (2.2) with respect to τ_j, $\frac{\partial}{\partial \tau_j}$ denoted by a dot, $\frac{\partial}{\partial x}$ by a prime we find

$$(\dot{\theta}'_{d+1}\theta_{d-1} - \dot{\theta}_{d+1}\theta'_{d-1}) + (\theta'_{d+1}\dot{\theta}_{d-1} - \theta_{d+1}\dot{\theta}'_{d-1}) = 2(2d + 1)\theta_d\dot{\theta}_d .$$

For $\tau_1 = \tau_2 = \cdots = \tau_d = 0$ this gives

$$\{(n_{d+1} - 2j + 1) - n_{d-1}\}\alpha_j(d+1) + \{n_{d+1} - (n_{d-1} - 2j + 1)\}\alpha_j(d-1) = 2(2d+1)\alpha_j(d)$$

or since $n_{d+1} - n_{d-1} = 2d + 1$

$$(d - j + 1)\alpha_j(d + 1) + (d + j)\alpha_j(d - 1) = (2d + 1)\alpha_j(d) .$$

This recursion formula for $\alpha_j(d)$, together with $\alpha_j(d) = 1$ for $d = j$ (by normalization) and $\alpha_j(d) = 0$ for $d > j$, determines $\alpha_j(d)$ uniquely. In fact one finds explicitly

$$\alpha_j(d) = \binom{d + j}{d - j} = \binom{d + j}{2j} \neq 0 \quad \text{for} \quad j = 1,2,\ldots,d$$

which proves the lemma.

This lemma has the following consequence: The polynomials $\theta_d(x + \tau_1,\tau_2,\ldots,\tau_d)$ are uniquely determined by the choice of the $\tau_1,\ldots,\tau_d$. Indeed, if

$$\theta_d(x + \tau_1,\tau_2,\ldots,\tau_d) = \theta_d(x + \hat{\tau}_1,\hat{\tau}_2,\ldots,\hat{\tau}_d)$$

for all x then the coefficients $\sigma_j(\tau) = \sigma_j(\hat{\tau})$ agree, which implies by the above lemma that $\tau_k = \hat{\tau}_k$. This remark implies that the representation (1.3) of $u \in M_d$ is one to one.

Remark 3: For $\tau_2 = \tau_3 = \cdots = \tau_{d-2} = 0$ one can easily compute θ_d explicitly and finds for $d \geq 3$

$$\theta_d = x^{n_{d-4}}\{x^{4d-6} + (2d - 1)\tau_{d-1}x^{2d-3} - \frac{2d - 1}{2d - 5}\tau_{d-1}^2 + \tau_d x^{2d-5}\} .$$

This formula contains as special cases those of [1], proposition 2 and 3, Section 5.

§3. A deformation problem for the modified Korteweg-de Vries equation.

In [5] Lax related the Korteweg-de Vries equation - as well as its higher analogues - to the iso-spectral deformation of the operator

$$L = -D^2 + u \tag{3.1}$$

where $u = u(x)$ is a C^∞-function. We will describe an analogue problem for the first order operator

$$A = D - v \tag{3.2}$$

with $v = v(x)$.

For motivation of the following we consider at first a bounded linear operator A in a Hilbert space and call A equivalent to A_0 if there exist two unitary operators U_1, U_2 such that

$$U_1^{-1} A U_2 = A_0 \ . \tag{3.3}$$

Clearly the invariants of this equivalence relation are the spectral invariants of A^*A or of AA^*.

We ask for deformations $A(t)$ of $A_0 = A(0)$ which remain in the same equivalence class. Assuming that $U_1 = U_1(t)$, $U_2 = U_2(t)$ are defined through differential equations

$$\dot{U}_j = B_j U_j; \ U_j(0) = I; \ j = 1,2$$

with skew Hermitian B_j, we obtain by differentiation of (3.3)

$$U_1^{-1}(\dot{A} - B_1 A + A B_2) U_2 = 0$$

or

$$\dot{A} = B_1 A - A B_2 \ . \tag{3.4}$$

We apply this consideration to $A = D - v$, $v = v(x,t)$ and choose

$$B_j = D^3 + b_j D + D b_j$$

as skew Hermitian operator. We now consider (3.4) as a <u>formal</u> relation for differential operators. The left hand side of (3.4) is a multiplication operator, namely multiplication by $-v_t$ and b_1, b_2 have to be so determined that in $B_1A - AB_2$ the coefficients of D^4, D^3, D^2, D vanish. The first two coefficients vanish automatically while the other two are

$$-3v' + 2(b_1 - b_2)$$
$$-3v'' + b_1' - 3b_2' - 2(b_1 - b_2)v \,.$$

Setting these expressions equal to zero yields two linear equations for b_1, b_2 with the solution

$$b_1 = -\frac{3}{4}(-v' + v^2) + c$$
$$b_2 = -\frac{3}{4}(v' + v^2) + c$$

with an arbitrary integration constant c. This constant reflects the trivial solution of (4) with $B_j = 2cD$, $v_t = v_x$. Therefore we set $c = 0$ and obtain from (3.4) a partial differential equation for v which is computed to be

$$v_t = \frac{1}{4} v''' - \frac{3}{2} v^2 v' \,. \tag{3.5}$$

This is the so-called modified Korteweg-de Vries equation which was used by Miura [7] in his derivation of the conservation laws for the KdV equation. For this purpose he showed that the function

$$u = v^2 + v' \tag{3.6}$$

satisfies

$$u_t = \frac{1}{4} u''' - \frac{3}{2} uu' \,, \tag{3.7}$$

if v is a solution of (3.5). This remarkable fact has a natural explanation in the following observation:

If $u = v' + v^2$ then the operator (3.1) can be factored as

$$L = A^* A \tag{3.8}$$

where

$$A^* = -D - v$$

is the formal adjoint of A. Moreover, the deformation equation (3.4) gives rise to a deformation equation for L

(3.9) $\dot{L} = \dot{A}^*A + A^*\dot{A} = -A^*B_1A + B_2A^*A + A^*B_1A - A^*AB_2 = [B_2,L]$;

here we used that $B_j^* = -B_j$, $B_j = B_j(v)$. This is precisely the deformation problem

(3.10) $\dot{L} = [B,L]; \quad B = B(u) = D^3 + (b(u)D + Db(u))$

studied by Lax [5], which leads to (3.7) and $B_2(v) = B(v' + v^2)$, provided the arbitrary constant is normalized appropriately. This shows then that any solutions v of (3.5) gives rise to a solution u of (3.7) via (3.6). This appears as a consequence of the factorization (3.8).

If instead we consider the operator

$$\tilde{L} = AA^* = -D^2 + \tilde{u}$$

with $\tilde{u} = -v' + v^2$, then clearly we find analogously to (3.9)

$$\dot{\tilde{L}} = [B_1,\tilde{L}]$$

and $B_1(v) = B(\tilde{u}) = B(-v' + v^2)$, where $B(\tilde{u})$ again denotes the third order operator obtained by Lax. Moreover, $\tilde{u} = -v' + v^2$ is automatically a solution of (3.7). The duality map $u \to \tilde{u}$ which takes solutions of (3.7) again into solutions of the same equation, a so-called Bäcklund transformation, is here related to the deformation of two products AA^*, A^*A, i.e. to exchanging A and A^*.

These considerations can be generalized to the higher Korteweg-de Vries operators X_j by considering real skew Hermitian operators B_1, B_2 of degree $2j - 1$. This leads to operators $B_1(v)$, $B_2(v)$ expressible with polynomial coefficients in $v, v', \ldots$ (this is a difficult point) which are of the form

$$B_1(v) = B(-v' + v^2)$$
$$B_2(v) = B(v' + v^2)$$

where $B = B(u)$ corresponds to the iso-spectral deformation of $L = A^*A$, i.e. $\dot{L} = [B,L]$ associated with the X_j-flow. For the explicit representation of $B(u)$ we refer to [6].

We will not present the proof of this statement since we find it easier to introduce the X_j via a recursion formula and then verify directly their invariance under the Bäcklund transformation. This will be done in Section 5.

§4. The construction of the rational potentials.

From the factorization

$$L = A^*A, \quad \tilde{L} = AA^*$$

one reads off that

$$AL = \tilde{L}A . \tag{4.1}$$

This identity shows that

$$(L - \lambda)\varphi = 0 \quad \text{implies} \quad (\tilde{L} - \lambda)A\varphi = 0 \tag{4.2}$$

which allows us to construct the solutions of $(\tilde{L} - \lambda)\tilde{\varphi} = 0$ from those of $(L - \lambda)\varphi = 0$.

The mapping $u \to \tilde{u}$ is an involution, i.e. $\tilde{u}$ is mapped back into u. The following is closely related to [2]. However, if one observes that the factorization

$$L = A^*A; \qquad u = v' + v^2$$

depends on the choice of v one can reach new potentials by repeated application of this procedure. The various choices of v can be easily described by the solutions φ of

$$L\varphi = 0, \; v = \frac{\varphi'}{\varphi} .$$

Indeed with this choice of v one has $L = A^*A$ for $A = D - v$; as well as $A\varphi = 0$. In fact, we may write $A = \varphi D \varphi^{-1}$ and hence $A^* = -\varphi^{-1}D\varphi$.

From $A\varphi = 0$ one has $A^*\varphi^{-1} = 0$ and $\tilde{L} = -D^2 + \tilde{u}$, $\tilde{u} = u - 2v'$ satisfies

$$\tilde{L}\varphi^{-1} = 0 .$$

Instead of using φ^{-1} and $\tilde{v} = -\frac{\varphi'}{\varphi}$ as the basis for the factorization of $\tilde{L} = \tilde{A}^*\tilde{A}$ we pick any other solution $\tilde{\varphi}$ of $\tilde{L}\tilde{\varphi} = 0$. Since the Wronskian of $\tilde{\varphi}$ and φ^{-1} is a constant we determine $\tilde{\varphi}$ as a solution of

$$[\tilde{\varphi}, \varphi^{-1}] = \text{const} \neq 0 \tag{4.3}$$

where $[a,b] = a'b - ab'$ denotes the Wronskian determinant. Then

$$\tilde{v} = \frac{\tilde{\varphi}'}{\tilde{\varphi}} , \quad \tilde{u} = u - 2v'$$

gives rise to a new potential $\tilde{\tilde{u}} = \tilde{u} - 2\tilde{v}'$. The solution of (4.3) introduces one integration constant at this step.

This simple procedure, started with the potential $u = 0$ yields the rational potentials: We set $u_0 = 0$, $\varphi_0 = x$ and construct a sequence of potentials u_k inductively as follows. We construct φ_k from

$$[\varphi_k, \varphi_{k-1}^{-1}] = \text{const} \neq 0 \quad \text{for} \quad k = 1,2,\ldots \tag{4.4}$$

and set for $k \geq 1$

$$u_k = \frac{\varphi_k''}{\varphi_k}. \tag{4.5}$$

Since this transformation which maps u_k into u_{k+1} takes solution of the KdV equations again to solutions of the KdV equations, we will obtain in u_k a k-parameter family of solutions of these equations since every step introduces one integration constant. It is surprising, that this procedure gives rise to rational functions u_k, in fact, the most general rational solutions of the KdV equations, aside from the trivial one $u = -\frac{3x}{t}$ of (1.1). The following lemma expresses the result of this construction (4.4), (4.5) in terms of the θ_k.

Lemma 2: If θ_k denotes the polynomials of §2 then the rational functions

$$\left.\begin{aligned} u_k &= -2\left(\frac{\theta_k'}{\theta_k}\right)' = -2(\log \theta_k)'' \\ \varphi_k &= \frac{\theta_{k+1}}{\theta_k}; \quad v_k = \frac{\varphi_k'}{\varphi_k} \end{aligned}\right\} k = 0,1,\ldots \tag{4.6}$$

satisfy

$$[\varphi_k, \varphi_{k-1}^{-1}] = 2k+1$$

$$(-D^2 + u_k)\varphi_k = 0 \tag{4.7}$$

$$\begin{cases} u_k = v_k' + v_k^2 \\ u_{k+1} = -v_k' + v_k^2 . \end{cases} \tag{4.8}$$

Proof: For $k = 0$ the definition of $\theta_0 = 1$, $\theta_1 = x$ leads to $u_0 = 0$, $\varphi_0 = x$ in agreement with our requirement. The recursion formula (2.2) leads to

$$[\varphi_k, \varphi_{k-1}^{-1}] = \left[\frac{\theta_{k+1}}{\theta_k}, \frac{\theta_{k-1}}{\theta_k}\right] = 2k + 1 .$$

The definition of u_k, φ_k yield

$$u_k = -2\left(\frac{\theta_k'}{\theta_k}\right)' = -2\,\frac{\theta_k''}{\theta_k} + 2\left(\frac{\theta_k'}{\theta_k}\right)^2$$

$$\frac{\varphi_k''}{\varphi_k} = \frac{\theta_{k+1}''}{\theta_{k+1}} - \frac{\theta_k''}{\theta_k} - 2\,\frac{\theta_k'}{\theta_k}\left(\frac{\theta_{k+1}'}{\theta_{k+1}} - \frac{\theta_k'}{\theta_k}\right)$$

so that

$$\frac{\varphi_k''}{\varphi_k} - u_k = \frac{\theta_{k+1}''}{\theta_{k+1}} + \frac{\theta_k''}{\theta_k} - 2\,\frac{\theta_k'}{\theta_k}\,\frac{\theta_{k+1}'}{\theta_{k+1}} = \frac{\{\theta_k, \theta_{k+1}\}}{\theta_k\,\theta_{k+1}}$$

which vanishes by (2.3).

The identity

$$v_k' = \left(\frac{\varphi_k'}{\varphi_k}\right)' = \frac{\varphi_k''}{\varphi_k} - \left(\frac{\varphi_k'}{\varphi_k}\right)^2 = u_k - v_k^2$$

proves one of the last relations. Finally $\theta_{k+1} = \theta_k \varphi_k$ implies

$$u_{k+1} = -2(\log(\theta_k \varphi_k))'' = u_k - 2\left(\frac{\varphi_k'}{\varphi_k}\right)'$$

$$= u_k - 2v_k' = -\,v_k' + v_k^2$$

proving the lemma.

We summarize: The relations (4.8) show that u_{k+1} is obtained from u_k by the Bäcklund transformation of § 3 while (4.6) expresses these potentials in terms of the polynomials θ_k. The equation (4.7) implies that

$$L_k \varphi = (-D^2 + u_k)\varphi = 0$$

has one solution $\varphi = \varphi_k = \dfrac{\theta_{k+1}}{\theta_k}$. The Wronskian relation shows that

$\varphi = \varphi_{k-1}^{-1} = \dfrac{\theta_{k-1}}{\theta_k}$ is another solution. Thus the most general solution is given by

$$\varphi = (c_1\theta_{k-1} + c_2\theta_{k+1})\theta_k^{-1} .$$

It is interesting to note that also the solutions of the equation

$$(L_k - \lambda)\psi = 0$$

are the product of rational functions in x and $e^{\pm\omega x}$, $\lambda = -\omega^2$. Indeed by (4.2) we see: If ψ solves the above equation then

$$\tilde{\psi} = (D - v_k)\psi = \left(D - \frac{\varphi_k'}{\varphi_k}\right)\psi = A_k\psi$$

is a solution of $(L_{k+1} - \lambda)\psi = 0$. For $k = 0$ we have $\psi = c_1e^{\omega x} + c_2e^{-\omega x}$, hence

$$\psi = A_k \dots A_2A_1(e^{\pm\omega x})$$

is a basis for $(L_{k+1}-\lambda)\psi = 0$. This is essentially contained in [2], in a different guise.

As the special example we mention the case $\tau_2 = \tau_3 = \cdots = \tau_k = 0$, i.e. $\theta_k(x) = x^{n_k}$ or $u_k = \dfrac{2n_k}{x^2}$. The equation

$$D^2\psi + \left(\lambda - \frac{k(k+1)}{x^2}\right)\psi = 0$$

has the solutions $\psi = \sqrt{x}\, J_{\pm(k+\frac{1}{2})}(i\omega x)$ with the Bessel functions J_ν, which for integer $\nu - \frac{1}{2}$ are indeed elementary functions of the indicated nature.

§5. The KdV-flows.

Following Lenard we define the KdV vector fields X_k and the Hamiltonian H_k

$$X_k(u) = D \frac{\delta H_k}{\delta u} \tag{5.1}$$

recursively by $X_1 = D$, or $X_1(u) = u'$, and

$$X_k = (uD + Du - \tfrac{1}{2} D^3) \frac{\delta H_{k-1}}{\delta u} \qquad k = 1,2,\ldots \tag{5.2}$$

or formally

$$X_k = (u + DuD^{-1} - \tfrac{1}{2} D^2) X_{k-1} = RX_{k-1} , \tag{5.3}$$

where

$$R = u + DuD^{-1} - \tfrac{1}{2} D^2 . \tag{5.4}$$

This definition requires the verification that at each step X_{k-1} can be written as the derivative of $\frac{\delta H_{k-1}}{\delta u}$, which is expressed as a polynomial in u and finitely many of its derivatives. For the proof of this fact, see e.g. [3], [6]. Also X_k is defined up to an arbitrary constant which is normalized by the requirement that $X_k(u)$ is an "isobaric" polynomial in $u,u',u'',\ldots$ of degree $2k + 1$, which means that every term

$$u^{\alpha_0}(Du)^{\alpha_1}(D^2u)^{\alpha_2} \ldots$$

in $X_k(u)$ satisfies $\sum_{\nu \geq 0} (2 + \nu)\alpha_\nu = 2k + 1$; i.e. u is assigned the weight 2 and each derivative the weight 1. (This weighting is consistent with the fact that in $L = -D^2 + u$ multiplication with u and D^2 are on the same footing.)

For $k = 2$, for example, we find

$$X_2(u) = (uD + Du - \tfrac{1}{2} D^3)u = 3uu' - \tfrac{1}{2} u''' ,$$

which agrees with (3.7) up to an unessential factor -2.

Similarly we introduce a sequence of vector fields $v_t = Y_k(v)$ with $-\frac{1}{2}Y_2(v)$ corresponding to (3.5). The requirement is that every solution of $v_t = Y_k(v)$ gives rise, via $u = v' + v^2$, to a solution of $u_t = X_k(u)$. This requirement leads to the following recursive definition:

Let S be the formal operator introduced by Olver in [10], namely

$$S = 2v^2 + 2v'D^{-1}v - \frac{1}{2}D^2 \tag{5.5}$$

and set

$$\begin{cases} Y_1(v) = v' \\ Y_k(v) = SY_{k-1}(v) \quad \text{for} \quad k = 1,2,\dots \quad . \end{cases} \tag{5.6}$$

Again it is important to show that $Y_k(v)$ can be defined as polynomials in $v, v', \dots,$ which we will do presently. Secondly, the definition is made unique by requiring that $Y_k(v)$ are isobaric polynomials of degree $2k$ if v and $\frac{\partial}{\partial x}$ are assigned the weight 1 each. For example, for $Y_2(v)$ we find

$$Y_2(v) = (2v^2 + 2v'D^{-1}v - \frac{1}{2}D^2)v' = 3v^2v' - \frac{1}{2}v'''$$

which agrees with (3.5) up to the factor -2.

Lemma 3: i) The $Y_k(v)$ are uniquely defined as isobaric polynomials of degree $2k$ by the recursion formula (5.6) and ii) satisfy

$$(2v + D)Y_k(v) = X_k(v^2 + v') \tag{5.7}$$

Moreover

$$Y_k(-v) = -Y_k(v).$$

Proof: We proceed by induction. For $k = 1$ the above statement is evident. We assume it holds for k and verify it for $k+1$. First it is to be shown that

$$SY_k = (2v^2 + 2v'D^{-1}v - \frac{1}{2}D^2)Y_k(v)$$

can be defined as an isobaric polynomial in v and its derivatives. For this it is sufficient to verify this for $D^{-1}(vY_k(v))$ or for

$$D^{-1}(2v + D)Y_k(v) = 2D^{-1}(vY_k) + Y_k(v).$$

But by the induction hypothesis (see (5.7)) we have

$$(2v + D)Y_k = X_k(u) = D\,\frac{\delta H_k(u)}{\delta u}$$

with $u = v' + v^2$, and therefore

$$D^{-1}(2v + D)Y_k = \frac{\delta H_k}{\delta u}$$

is indeed a polynomial in $u = v' + v^2$ and its derivatives. Of course, one could add an arbitrary constant to $D^{-1}(2v + D)Y_k$ but the choice is unique by requiring that $D^{-1}(2v + D)Y_k(v)$ be isobaric of degree $2k$ in $v, v', \ldots$. Note that $\frac{\delta H_k}{\delta u}$ is isobaric of degree $2k$ in $u, u', \ldots$ when $u = v' + v^2$, and by induction it follows that

$$Y_{k+1} = SY_k = (2v^2 - v' - \tfrac{1}{2}D^2)Y_k(v) + v'\,\frac{\delta H_k}{\delta u}$$

is isobaric of degree $2k + 2$ in $v, v', \ldots$.

To prove (5.7) we observe the intertwining identity

$$R(2v + D) = (2v + D)S$$

for S which is readily verified. It implies

$$\begin{aligned}(2v + D)Y_{k+1}(v) &= (2v + D)S\,Y_k(v) = R(2v + D)Y_k(v)\\ &= RX_k(v' + v^2) = X_{k+1}(v' + v^2)\ ,\end{aligned}$$

where the isobaric character of the X_k, Y_k is used implicitly to fix the integration constants. This completes the induction.

Thus the polynomials $Y_k(v)$ $(k \geq 1)$ in $v, v', v'', \ldots$ are uniquely defined. Moreover, they are odd in v, i.e. satisfy

$$Y_k(-v) = -Y_k(v)\ .$$

Indeed this follows immediately from the recursion (5.6) since S is even in v and $Y_1(v) = v'$ is odd, which completes the lemma.

Next we show that X_k leaves the manifold $u_d = -2(\log \theta_d)''$ invariant. This is the contents of

Theorem 1: There exists a unique choice of rational functions $\gamma_{kj}(\tau_2,\ldots,\tau_j)$ and differential operators

$$\Gamma_k = \sum_{j=1}^{\infty} \gamma_{kj} \frac{\partial}{\partial \tau_j}$$

such that for $d = 0,1,2,\ldots$

(5.8) $$X_k(u_d) = \Gamma_k u_d$$

and

(5.9) $$Y_k(v_d) = \Gamma_k v_d \quad \text{where} \quad v_d = \frac{\theta'_{d+1}}{\theta_{d+1}} - \frac{\theta'_d}{\theta_d} .$$

(Since u_d, v_d depend only on finitely many variables the sum breaks off.)

In other words, if the τ_j satisfy

$$\dot{\tau}_j = \gamma_{kj}(\tau_2,\ldots,\tau_j) \qquad j \le d$$

then $u = u_d(x + \tau_1, \tau_2, \ldots, \tau_d)$ solves the equation

$$\frac{\partial u}{\partial t_k} = X_k(u) .$$

Proof: We proceed by induction on d. For $d = 0$ we have $u_0 = 0$ and therefore $X_k(u_0) = 0$. Assume next that $\gamma_{kj} = \gamma_{kj}(\tau_2,\ldots,\tau_j)$ for $j = 1,2,\ldots,d$ have been determined such that (5.8) holds. Then we conclude from (5.7) and $u_d = v'_d + v_d^2$ with $v_d = \frac{\varphi'_d}{\varphi_d}$, $\varphi_d = \frac{\theta_{d+1}}{\theta_d}$ that

$$(2v_d + D)Y_k(v_d) = X_k(u_d) = \Gamma_k u_d = (2v_d + D)\Gamma_k v_d$$

or

$$(2v_d + D)(Y_k(v_d) - \Gamma_k v_d) = 0 .$$

Since $\psi = \varphi_d^{-2}$ is a solution of $(2v_d + D)\psi = 0$ we conclude that

(5.10) $$Y_k(v_d) - \Gamma_k v_d = c\varphi_d^{-2}$$

with c being a rational function of $\tau_2,\ldots,\tau_{d+1}$. On the other hand v_d depends on τ_{d+1} and in

$$\Gamma_k v_d = \sum_{j\le d} \gamma_{kj} \frac{\partial v_d}{\partial \tau_j} + \gamma_{k,d+1} \cdot \frac{\partial v_d}{\partial \tau_{d+1}} \tag{5.11}$$

the coefficient $\gamma_{k,d+1}$ can be uniquely determined so that $c = 0$. Indeed, $v_d = \frac{\varphi_d'}{\varphi_d} = \frac{\theta_{d+1}'}{\theta_{d+1}} - \frac{\theta_d'}{\theta_d}$ where θ_d is independent of $\tau = \tau_{d+1}$ while $\frac{d\theta_{d+1}}{d\tau} = \theta_{d-1}$. Hence, by (2.2)

$$\frac{\partial v_d}{\partial \tau_{d+1}} = \left(\frac{\theta_{d-1}}{\theta_{d+1}}\right)' = -(2d+1)\frac{\theta_d^2}{\theta_{d+1}^2} = -(2d+1)\varphi_d^{-2}$$

and the coefficient of $\gamma_{k,d+1}$ in (5.11) is $-(2d+1)\varphi_d^{-2}$; thus we have from (5.10)

$$\varphi_d^2\{Y_k(v_d) - \sum_{j\le d} \gamma_{kj} \frac{\partial v_d}{\partial \tau_j}\} = -(+2d+1)\gamma_{k,d+1} + c \quad .$$

By appropriate choice of $\gamma_{k,d+1}$, as rational function of $\tau_1,\dots,\tau_{d+1}$, we obtain $c = 0$, i.e.

$$Y_k(v_d) = \Gamma_k v_d$$

as claimed (5.9). This determines $\gamma_{k,d+1} = \gamma_{k,d+1}(\tau_2,\dots,\tau_{d+1})$ uniquely.

Using that $Y_k(-v) = -Y_k(v)$ we conclude from (5.7)

$$\begin{aligned} X_k(u_{d+1}) = X_k(-v_d' + v_d^2) &= (-2v_d + D)Y_k(-v_d) \\ &= (2v_d - D)Y_k(v_d) \\ &= (2v_d - D)\Gamma_k v_d \\ &= \Gamma_k(v_d^2 - v_d') = \Gamma_k u_{d+1} \end{aligned}$$

which completes the induction and the proof.

<u>Lemma 4</u>: The $\gamma_{kj} = \gamma_{kj}(\tau)$ are polynomials of $\tau_2,\tau_3,\dots$ with rational coefficients. If we assign τ_m the weight $2m-1$ the γ_{kj} are isobaric of weight $2(j-k)$. In particular, they depend on τ_ℓ with $\ell \le [j-k] \le j-1$ only. Moreover, γ_{kk} is a non-vanishing constant.

Proof: First we express the differential equation $u_t = X_k(u)$ via $u = -2(\log \theta)''$ in terms of

$$\theta = \theta_d = x^n + \sigma_1 x^{n-1} + \cdots + \sigma_n$$

with $n = n_d$. This expresses the differential equation in terms of the σ_j which finally are transferred to the τ_j via the corollary of lemma 1 of Section 2.

From

$$-2 \frac{\partial}{\partial t} D\left(\frac{\theta'}{\theta}\right) = u_t = X_k(u) = D \frac{\delta H_k}{\delta u}$$

we conclude that

$$\theta^{-2}(\dot{\theta}\theta' - \dot{\theta}'\theta) = \frac{1}{2} \frac{\delta H_k}{\delta u},$$

an integration constant being eliminated by the isobaric property. The right hand side depends on $u, u', \ldots,$ and we observe that

$$D^m u = \theta^{-m-2} P_m(\theta, \theta', \ldots)$$

with P_m being a polynomial in $\theta, \theta', \ldots$. Since $\frac{\delta H_k}{\delta u}$ is isobaric of degree $2k$ we find that

$$\theta^{2k} \frac{1}{2} \frac{\delta H_k}{\delta u} = Q(\theta, \theta', \ldots)$$

is a polynomial in $\theta, \theta', \ldots$. Thus the differential equation takes the form

$$\theta^{2k-2}(\dot{\theta}\theta' - \dot{\theta}'\theta) = Q(\theta, \theta', \ldots) .$$

In this equation we compare coefficients of x. The term of highest power in x containing σ_j is

$$x^{(2k-2)n} x^{n-j} x^{n-1} (n - (n - j))\dot{\sigma}_j = x^{2kn-j-1} j\dot{\sigma}_j$$

i.e. comparing the coefficients of $x^{2kn-j-1}$ we find

$$j\dot{\sigma}_j = \sum_{m<j} a_m \dot{\sigma}_m + a_j$$

where $a_1, a_2, \ldots, a_j$ are polynomials in the σ. Thus we find

$$\dot{\sigma}_j = b_j(\sigma_1,\ldots,\sigma_d)$$

with polynomials b_j. Finally using the corollary of lemma 1 we express these differential equations in terms of $\tau_1,\tau_2,\ldots$ in the form

$$\dot{\tau}_j = \gamma_{kj}(\tau_1,\tau_2,\ldots) \; .$$

One readily checks the homogeneity of the γ_{kj} to be given by

$$\gamma_{kj}(\lambda\tau_1,\lambda^3\tau_2,\ldots) = \lambda^{2j-2k}\gamma_{kj} \; ,$$

by using an argument like in Remark 1, Section 2, observing the isobaric property of X_k. In particular, $\gamma_{kj} = 0$ for $j < k$ and γ_{kk} is a constant.

To evaluate this constant we use the fact that

$$u_d = \frac{2n_d}{x^2} \quad \text{for} \quad \tau_2 = \tau_3 = \cdots = \tau_d = 0$$

and for $d = k$

$$X_k u_k = c_k x^{-(2k+1)}$$

with a rational constant $c_k \neq 0$ as one computes from (5.3). Thus we have for $\tau_2 = \tau_3 = \cdots = \tau_k = 0$

$$0 \neq X_k u_k = \Gamma_k u_k = \sum_{j=1}^{k} \frac{\partial u_k}{\partial \tau_j}\gamma_{kj} = \frac{\partial u_k}{\partial \tau_k}\gamma_{kk} \; ,$$

as $\gamma_{kj} = 0$ for $j < k$. Hence $\gamma_{kk} \neq 0$ as was claimed.

Thus the differential equations induced by X_k are given by Γ_k, or equivalently by

$$\dot{\tau}_j = \gamma_{kj}(\tau_2,\tau_3,\ldots,\tau_{j-1}) \; ,$$

with isobaric polynomials γ_{kj}. Obviously the equations can be solved recursively as polynomials in t. In fact, more is true: There exists an isobaric birational transformation $\tau_j \to \tau_j^*$ such that for all $k \geq 1$ we have

$$\Gamma_k = \frac{\partial}{\partial \tau_k^*}$$

i.e. the above differential equation reduces to

$$\Gamma_k \tau_j^* = \delta_{kj}$$

and τ_k^* can be interpreted as the time variable for the flow X_k. This is a simple consequence of the fact that the X_k - and hence the Γ_k - commute. We recall that the τ_k were introduced by an artificial normalization and these parameters are actually defined only up to the group of birational transformations which commute with $\tau_j \to \lambda^{2j-1}\tau_j$. Now we make the unique choice of the parameters so that they are adapted to the KdV flows.

Theorem 2: There exists a unique birational transformation

$$\tau_j^* = a_j \tau_j + g_j(\tau_2, \tau_3, \ldots, \tau_{j-1})$$

with g_j being isobaric polynomials

$$g_j(\lambda^3\tau_2, \lambda^5\tau_3, \ldots, \lambda^{2j-3}\tau_{j-1}) = \lambda^{2j-1} g_j(\tau_2, \ldots, \tau_{j-1})$$

with rational coefficients and a rational number $a_j \neq 0$ such that

$$\Gamma_j = \frac{\partial}{\partial \tau_j^*} .$$

Corollary: If $u = u_d(\tau_1, \tau_2, \ldots, \tau_d)$ is expressed in terms of τ^* as $u = u_d^*(\tau_1^*, \ldots, \tau_d^*)$ then the function

$$u = u_d^*(\tau_1^* + x, \tau_2^*, \ldots, \tau_d^*)$$

is a solution of $\frac{\partial u}{\partial \tau_k^*} = X_k u$.

Proof: We proceed by induction as follows: Changing the notation of the τ_j also we assume that for some $k \geq 1$ we have*

$$\Gamma_j = \partial_{\tau_j} + \gamma_{jk}\partial_{\tau_k}, \qquad 1 \leq j \leq k-1 ,$$

$$\Gamma_k = \gamma_{kk}\partial_{\tau_k}$$

up to terms ∂_{τ_m} for $m > k$, which will be suppressed. For $k = 1$ this

*We abbreviate $\frac{\partial}{\partial \tau}$ by ∂_τ.

is trivially the case and we will construct a transformation

$$(5.12)\quad \begin{cases} \tau_j^* = \tau_j & (j \neq k) \\ \tau_k^* = a_k\tau_k + g_k(\tau_2,\ldots,\tau_{k-1}) \end{cases}$$

such that

$$\Gamma_j = \partial_{\tau_j^*} + 0\cdot\partial_{\tau_k^*}$$

$$\Gamma_k = \partial_{\tau_k^*}$$

up to terms $\partial_{\tau_m^*}$ with $m > k$. This is obviously sufficient for the proof, if we also verify that the g_k and a_k have the required properties.

To carry out this induction step it is convenient to break up (5.12) into k steps and effectively make a second induction. First we achieve by $\tau_k^* = a_k\tau_k$ that $\Gamma_k = \gamma_{kk}a_k\partial_{\tau_k^*} = \partial_{\tau_k^*}$ by setting $a_k = \gamma_{kk}^{-1}$, a rational number. Assume now that we already achieved

$$\gamma_{jk} = 0 \quad \text{for} \quad s < j < k .$$

Then for $s < j < k$, using the commuting of the X_k, hence Γ_k, we compute

$$0 = \Gamma_j\Gamma_s - \Gamma_s\Gamma_j = \frac{\partial\gamma_{sk}}{\partial\tau_j}\,\partial_{\tau_k}$$

hence γ_{sk} depends on $\tau_2,\tau_3,\ldots,\tau_s$ only. Therefore we construct a transformation

$$\tau_k^* = \tau_k + g(\tau_2,\ldots,\tau_s)$$

so that

$$\Gamma_j = \partial_{\tau_j} \quad \text{for} \quad s < j \leq k$$

and

$$\Gamma_s = \partial_{\tau_s^*} + \left(\frac{\partial g}{\partial\tau_s} + \gamma_{sk}\right)\partial_{\tau_k^*} ,$$

while Γ_m, $1 \leq m \leq s - 1$, maintains its inductively assumed form.

Thus, if we determine the polynomial g such that

$$\frac{\partial g}{\partial \tau_s} + \gamma_{sk} = 0$$

then we have effectively replaced γ_{sk} by 0. The choice of g is unique if we require it to be isobaric. After finitely many steps we achieve

$$\Gamma_j = \partial_{\tau_j^*}, \quad 1 \le j \le k ,$$

up to terms ∂_{τ_m} with $m > k$. This completes the induction argument. The isobaric character of the g_j follows readily from that of the γ_{kj}. This completes the proof of Theorem 2.

In conclusion we remark that the manifold M_d of rational function $u_d = -2(\log \theta_d)''$ agrees with the manifold considered in [1]. Therefore the roots $x_1, x_2, \ldots, x_n$, $n = n_d$ of θ_d:

$$\theta_d = \prod_{j=1}^{n} (x - x_j)$$

i.e. the poles of

$$u_d = 2 \sum_{j=1}^{n} (x - x_j)^{-2}$$

satisfy, by the derivations in [1] the algebraic equations (1.4). This fact could also be verified directly.

§6. Explicit representations for the θ_d.

If one makes use of Crum's formulae (see [2]) one can represent the polynomials θ_d as well as the transformation $A_{d-1}A_{d-2} \cdots A_0$ of §4 in terms of Wronskians in explicit form. In order to do this we define the Wronskian of k functions $\psi_1,\psi_2,\ldots,\psi_k$ as

$$W_k = W(\psi_1,\psi_2,\ldots,\psi_k) = \det(D^{i-1}\psi_j) \qquad i,j = 1,2,\ldots,k .$$

For abbreviation we also set

$$W_k(\chi) = W(\psi_1,\psi_2,\ldots,\psi_k,\chi)$$

with another smooth function χ. Then one has Jacobi's identity

$$(6.1) \qquad [W_k(\chi),W_{k+1}] = W_{k+1}(\chi)W_k \quad \text{for} \quad k = 1,2,\ldots .$$

This is readily verified. The left hand side is a linear differential operator of order $k+1$ in χ which clearly vanishes for $\chi = \psi_1,\psi_2,\ldots,\psi_k$ as well as for $\chi = \psi_{k+1}$. Thus, if we assume that the $\psi_1,\psi_2,\ldots,\psi_{k+1}$ are linearly independent, the left hand side must be a multiple of $W_{k+1}(\chi)$. Comparing the highest coefficient one obtains (6.1).

We apply the above definitions to a system ψ_j satisfying $\psi_0 = 0$, $\psi_1 = x$ and

$$(6.2) \qquad \psi_j'' = \psi_{j-1}, \qquad j = 1,2,\ldots,k .$$

Then one verifies, for $\chi = 1$ and setting $W_0 = 1$ that

$$(6.3) \qquad W_k(1) = (-1)^k W_{k-1} \quad \text{for} \quad k = 1,2,\ldots .$$

To prove this we write

$$W_k(1) = W(\psi_1,\psi_2,\ldots,\psi_k,1) = (-1)^k W(1,\psi_1,\ldots,\psi_k)$$

and since $\psi_1 = x$ the last expression reduces to

$$W_k(1) = (-1)^k W(\psi_2'',\psi_3'',\ldots,\psi_k'')$$

and by (6.2) to (6.3).

Thus setting $\chi = 1$ in (6.1) and using (6.3) we find

(6.4) $$[W_{k+1}, W_{k-1}] = W_k^2 \quad \text{for} \quad k = 1,2,\ldots$$

and

$$W_0 = 1, \; W_1 = \psi_1 .$$

Thus if we set $\psi_1 = x$ we see that θ_k and W_k differ only by a multiplicative factor:

(6.5) $$\theta_k = \mu_k W_k$$

where one determines

$$\mu_k = 1^k \cdot 3^{k-1} \cdot 5^{k-2} \ldots (2k-1)^1 = \prod_{j=1}^{k} (2k-2j+1)^j .$$

But this factor is unessential for the following. The choice $\psi_1 = x$ and (6.2) leads to

(6.6) $$\psi_j = \frac{x^{2j-1}}{(2j-1)!} + \sum_{i=0}^{j-2} \rho_{j-i} \frac{x^{2i}}{(2i)!} .$$

Equivalently one can define the ψ_j by the generating function

$$\sum_{j=1}^{\infty} \psi_j s^{2j-1} = \sin h(sx) + \Big(\sum_{i=2}^{\infty} \rho_i s^{2i-1}\Big) \cos h(sx)$$

where $\rho_2, \rho_3, \ldots$ are arbitrary parameters.

With this choice of $\psi_1, \psi_2, \ldots$ the formula (6.5) gives the desired explicit representation of θ_k. The $\rho_2, \rho_3, \ldots$ are birationally and isobarically related to the $\tau_2, \tau_3, \ldots$ and one finds for the first few values $\tau_2 = -3\rho_2$, $\tau_3 = 45\rho_3$; $\mu_0 = \mu_1 = 1$, $\mu_2 = 3$, $\mu_3 = 45$.

Now we express the mapping $T_d = A_{d-1}A_{d-2} \cdots A_0$ with

(6.7) $$A_j = \varphi_j D \varphi_j^{-1}; \; \varphi_j = \frac{\theta_{j+1}}{\theta_j} = \frac{W_{j+1}}{W_j}$$

in terms of the Wronskians.

Lemma 5: The map

$$\chi \to T_d\chi = A_{d-1}A_{d-2} \cdots A_0\chi$$

has the alternate form

$$T_d = \frac{W_d(\chi)}{W_d} \quad \text{for } d = 1,2,\ldots .$$

Proof: This formula occurs in [2]. Clearly for $d = 1$ it is easily verified and it suffices to check

$$A_d \frac{W_d(\chi)}{W_d} = \frac{W_{d+1}(\chi)}{W_{d+1}} .$$

Indeed

$$A_d \frac{W_d(\chi)}{W_d} = \varphi_d D\left(\frac{W_d}{W_{d+1}} \frac{W_d(\chi)}{W_d}\right) = \varphi_d \frac{[W_d(\chi), W_{d+1}]}{W_{d+1}^2} ,$$

and using (6.4), (6.7) the induction step is verified.

In order to interpret the spectral properties of the operator $L = L_d = -D^2 + u_d$, $u_d = -2(\log \theta_d)''$, we choose the parameters so that the roots of θ_d, i.e. the poles of u_d lie off the real axis, which requires complex potentials. The spectral problem, as well as the inverse problem for such complex potentials were studied by Marčenko [13]. We are indebted to Marčenko for the following interpretation.

First it is clear on account of

$$LT_d = T_d(-D^2)$$

that for $\lambda = -\omega^2$ that the solutions of $(L - \lambda)\varphi = 0$ are linear combinations of

$$\psi_\pm = \frac{W_d(e^{\pm\omega x})}{W_d} = e^{\pm\omega x} R_\pm(x)$$

with rational $R_\pm(x)$. Thus $\lambda > 0$ gives continuous spectrum with reflection coefficient zero. One finds $[\psi_+, \psi_-] = 2\omega^{2d+1}$.

Secondly $\lambda = 0$ has an eigenspace of L^N of dimension $\geq \left[\frac{d+1}{2}\right]$ in the Hilbert space of the absolutely square integrable functions if $N \geq \left[\frac{d+1}{2}\right]$. In fact, from the intertwining relation

$$T_d(-D^2) = L_d T_d$$

and $D^2 \frac{x^{2v}}{(2v)!} = \frac{x^{2v-2}}{(2v-2)!}$ one finds that the rational functions

$$\phi_v = T_d\left(\frac{x^{2v}}{(2v)!}\right) = \frac{1}{(2v)!} \frac{W_d(x^{2v})}{W_d}$$

satisfy

$$L\phi_v + \phi_{v-1} = 0 \ .$$

We remark that $T_d(x^{2j-1})$ are linearly dependent on $\phi_0, \phi_1, \ldots, \phi_v$ but the ϕ_v are linearly independent, as

$$\phi_v \sim c_v x^{2v-d} \quad \text{for} \quad |x| \to \infty; \quad \text{with} \quad c_v \neq 0 \ .$$

Thus the $\phi_v \in L^2$ if $0 \leq v < \frac{d}{2}$ and $\lambda = 0$ is an eiganvalue of higher multiplicity. All this requires, of course, that the parameters $\tau_1, \ldots, \tau_d$ so chosen that no root of θ_d is real.

These considerations allow the full discussion of the spectral properties of these potentials.

References

1. Airault, H., McKean, H. P. and Moser, J., Rational and Elliptic Solutions of the Korteweg - deVries Equation and a related Many Body Problem, Comm. Pure Appl. Math., Vol. 30, 1977.
2. Crum, M. M., Associated Sturm-Liouville Systems, Quart. Journ. Math., Ser. 2, Vol. 6, 1955, 121-127.
3. Gardner, C. S., Greene, J. M., Kruskal, M. D. and Miura, R. M., Korteweg-deVries Equation and Generalizations VI, Methods for Exact Solution, Comm. Pure Appl. Math., Vol. 27, 1974, 97-133.
4. Its, A. R. and Matveev, V. B., Hill's operator with finitely many gaps, Funct. Anal. Appl., Vol. 9, No. 1, 1975, 69-70.
5. Lax, P. D., Integrals of Nonlinear Equations of Evolution and Solitary Waves, Comm. Pure Appl. Math., Vol. 21, 1968, 467-490.
6. McKean, H. P. and van Moerbeke, P., The Spectrum of Hill's Equation, Inv. Math., Vol. 30, 1975, 217-274.
7. Miura, R. M., Korteweg-deVries Equation and Generalization I. A Remarkable Explicit Nonlinear Transformation, Journ. Math. Phys., Vol. 9, 1968, 1202-1204.
8. Miyake, M., Shimizu, K. and Mugibayashi, N., Soliton Solutions of the Korteweg-deVries Equation, Journ. Phys. Soc. Japan, Vol. 37, 1974, 868-873.
9. Moses, H. E., A Solution of the Korteweg-deVries Equation in a Half-Space Bounded by a Wall, Journ. Math. Phys., Vol. 17, No. 1, 1976, 73-75.
10. Olver, P., Evolution Equations Possessing Infinitely many Symmetries, preprint, 1976.
11. Wadati, M., Sanuki, H. and Konno, K., Relationships among Inverse Method, Bäcklund Transformation and an Infinite Number of Conservation Laws, Progr. Theor. Phys., Vol. 53, No. 2, 1975, 419.
12. Deift, P., Application of a Commutation Formula, Duke Math. Journ., to appear.
13. Marčenko, V., Met. Sbornik. 52 (94), No. 2, 1960, 739-788.

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In [1] special classes of solutions of the Korteweg-de Vries equation

$$u_t = 3uu_x - \frac{1}{2} u_{xxx} = X_2 u$$

were studied, in particular, all those $u = u(x,t)$ which are rational functions of x for each value of t. It turns out that these solutions are rational functions of t as well and of very special structure. In this paper ~~we give~~ a new construction of these solutions with emphasis on their algebraic properties is given.

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